INDE 301 Engineering Economy

# Chapter 2 Factors: How time and interest affect money

- **Single Payment Factors**
  - Recall that $P$ dollars now are equivalent to $F$ dollars after $n$ time periods at an interest rate of $i$ per time period[1], where

$$F = P(1+i)^n .$$

  - Rewrite this as

$$F = P \times (F/P, i, n) \ ,$$

where $(F/P, i, n) = (1+i)^n$ is the "*F*/*P* factor."

  - In addition, this implies that

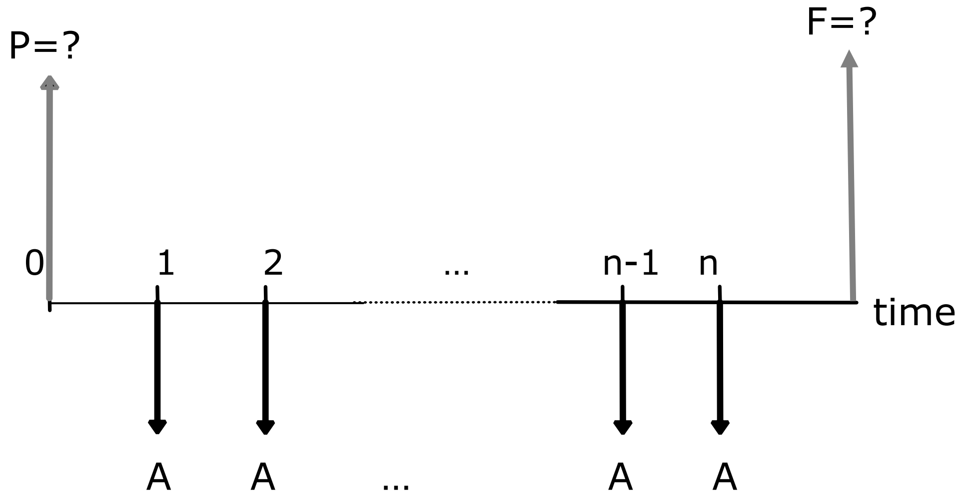
$$P = \frac{F}{(1+i)^n} = F \times (P/F, i, n),$$

where $(P/F, i, n) = \dfrac{1}{(1+i)^n}$ is the "*P/F* factor."

- **Tables and Spreadsheets**
  - The *P*/*F* and *F*/*P* factors, as well as other factors, are tabulated at the end of your textbook.
  - You may use these tables or the formulas that we will derive.
  - Spreadsheets (i.e., Excel) has built-in function for factors (or you can easily build your own functions) for the calculating the factors. Excel is very suited for practical interest calculations.

[1] Here and elsewhere, when the type of interest is not specified, assume it's *compound* interest.

- **Uniform Series Factors**
  - Suppose one will pay $A$ dollars every time period for n period starting with the end of period 1 (see figure below).




  - Then, this series of cash flows is now equivalent to

$$P=\frac{A}{1+i}+\frac{A}{(1+i)^2}+\ldots+\frac{A}{(1+i)^n}=\frac{A}{1+i}\sum_{j=0}^{n-1}\left(\frac{1}{1+i}\right)^j=\frac{A}{1+i}\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\left(\frac{1}{1+i}\right)}.$$

  - Upon simplification,

$$P=A\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]=\frac{A}{i}\left[1-\frac{1}{(1+i)^n}\right]=A\times(P/A,i,n).$$

  - $(P/A,i,n)=[(1+i)^n-1]/[i(1+i)^n]$ is the "P/A factor."
  - In addition,

$$A=P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]=P\times(A/P,i,n),$$

  - $(A/P,i,n)=i(1+i)^n/[(1+i)^n-1]$ is the "A/P factor."

- Finally, to find the future amount, $F$, equivalent to the uniform series of cash

$$F = P(1+i)^n = A\left[\frac{(1+i)^n - 1}{i}\right] = A \times (F/A, i, n),$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = \frac{F}{(1+i)^n}\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = F\left[\frac{i}{(1+i)^n - 1}\right]$$
$$= F \times (A/F, i, n),$$

- $(F/A, i, n) = [(1+i)^n - 1]/i$ is the "*F/A* factor."
- $(A/F, i, n) = i/[(1+i)^n - 1]$ is the "*A/F* factor."

- **Running amortization**
  - Amortization is the process of substituting a current payment $P$ for periodic payments of $A$ per period (e.g. car or home loan.)
  - One can view each amortization payment ($A$) as composed of two parts: (i) interest on running (outstanding) balance and (ii) partial repayment of principal.
  - This procedure is equivalent to re-amortizing the running balance every period over the remaining time horizon.
  - This is consistent with accounting practice.
  - E.g., consider a loan of $1,000 issued on Jan 1, 2006, to be paid back in equal monthly payments over 5 years at an interest rate of 12% per year compounded monthly.

- The monthly payment is

$$A = 1000\left[\frac{(0.01)(1+0.01)^{60}}{(1+0.01)^{60}-1}\right] = \$22.24 \ .$$

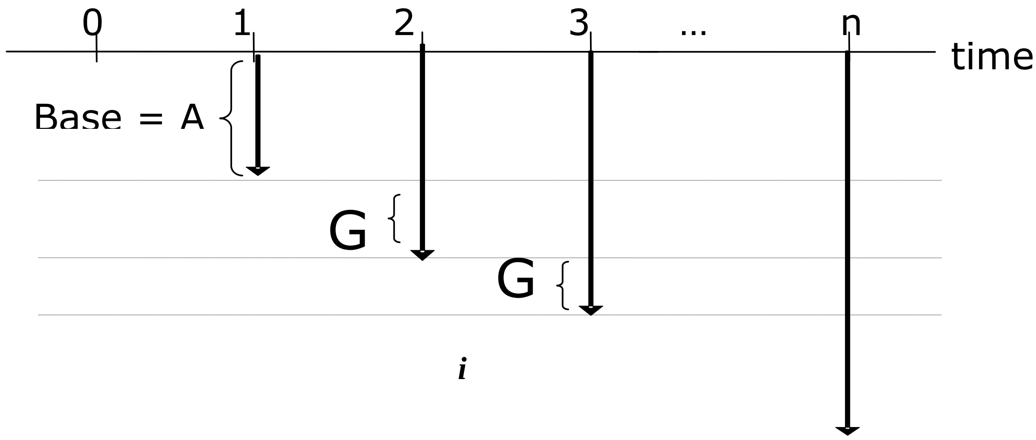
- Then, the outstanding balance on Feb 1, 2006 is $1,000 minus the monthly payment ($22.24) plus the monthly interest (0.01×1,000=$10), which gives $987.76.
- The (running) amortization of the $987.76 at 1-Mar-2006 over the remaining 59 months is

$$A = 987.67\left[\frac{(0.01)(1+0.01)^{59}}{(1+0.01)^{59}-1}\right] = \$22.24 \ .$$

- The (running) amortization of the $975.39 at 1-Apr-2006 over the remaining 58 months is also $22.24, and so on.

| **Date** | **Previous balance** | **Interest** | **Payment Received** | **New Balance** |
|---|---|---|---|---|
| 1-Jan-06 | | | | $1,000 |
| 1-Feb-06 | $1,000 | $10.00 | $22.24 | $987.76 |
| 1-Mar-06 | $987.76 | $9.88 | $22.24 | $975.39 |
| 1-Apr-06 | $975.39 | $9.75 | $22.24 | $962.90 |
| 1-May-06 | $962.90 | $9.63 | $22.24 | $950.28 |
| . | . | . | . | . |
| . | . | . | . | . |
| . | . | . | . | . |
| 1-Dec-10 | $43.83 | $0.44 | $22.24 | $22.02 |
| 1-Jan-11 | $22.02 | $0.22 | $22.24 | $0.00 |

- **Arithmetic Gradient Factors**




- In some cases cash flows increase by a fixed amount in each time period starting with period 2.
- Starting with a cash flow of *A* at the end of period 1, the cash flows increase by the gradient, *G*, in each period. That is, the cash flows are A, $A + G$, $A + 2G$, …, $A + nG$, in periods 1, 2, …, *n*.
- Then, at time 0, this series of cash flows is equivalent to

$$P = P_A + P_G \, ,$$

where $P_A$ is the equivalent at time zero of the series with uniform cash flows *A* per time period, $P_A = A\left[\dfrac{(1+i)^n - 1}{i(1+i)^n}\right]$ .

- $P_G$ is the equivalent at time zero of the arithmetic cash flow series with gradient G (i.e., the series having $G$, $2G$, …, $(n\!-\!1)G$ cash flows at the end of periods 2, 3, …, $n$.
- $P_G$ is evaluated as follows

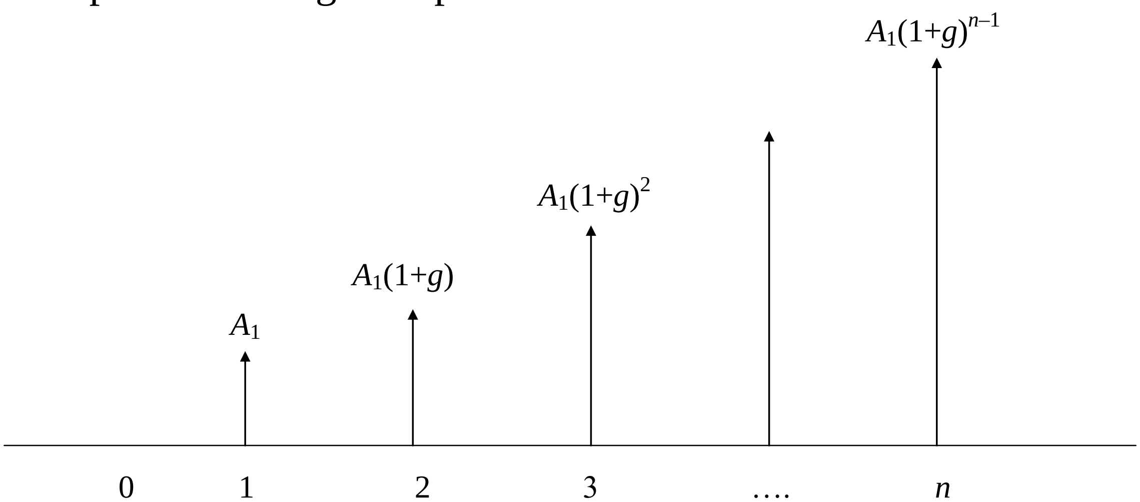
$$
\begin{aligned}
P_G &= \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \ldots + \frac{(n-2)G}{(1+i)^{n-1}} + \frac{(n-1)G}{(1+i)^n} \\
&= \frac{G}{1+i}\sum_{j=1}^{n-1}\frac{j}{(1+i)^j} = \frac{G}{1+i}\left(\frac{(1+i)-[1+i+(n-1)i]/(1+i)^{n-1}}{i^2}\right) \\
&= \frac{G}{i}\left(\frac{(1+i)^n-(1+ni)}{i(1+i)^n}\right) = \frac{G}{i}\left(\frac{(1+i)^n-1}{i(1+i)^n} - \frac{n}{(1+i)^{n-1}}\right) = G(P/G,i,n).
\end{aligned}
$$

- $(P/G,i,n) = \dfrac{G}{i}\left[\dfrac{(1+i)^n-1}{i(1+i)^n} - \dfrac{n}{(1+i)^n}\right]$ is the "P/G factor".
- In the above we have used the fact, that for $j$ and $m$ integers

$$\sum_{j=1}^{m} j/(1+y)^j = \frac{(1+y)-(1+y+my)/(1+y)^m}{y^2}.$$

- Finally, w define “A/G” and “F/G” factors.

- **Geometric Gradient Factors**
  - Suppose now that in a series of a cash flows the amounts increase (or decrease) by a fixed amount $(1+g)$, in each time period starting with period 2.




$$P_g = \frac{A_1}{1+i} + \frac{A_1(1+g)}{(1+i)^2} + \ldots + \frac{A_1(1+g)^{n-1}}{(1+i)^n} = \frac{A_1}{(1+i)} \sum_{j=0}^{n-1} \left( \frac{1+g}{1+i} \right)^j .$$

  - It follows that

$$P_g = \begin{cases} A_1 \left\{ \dfrac{1-[(1+g)/(1+i)]^n}{i-g} \right\}, & \text{if } i \neq g . \\ \dfrac{nA_1}{1+i}, & \text{if } i = g . \end{cases}$$

- **Summary of terminology**

*F/P* factor: Compound Amount Factor

*P/F* factor: Present Worth Factor

*P/A* factor : Uniform-Series Present Worth Factor

*A/P* factor: Capital Recovery Factor

*A/F* factor: Sinking Fund Factor

*F/A* factor: Uniform-Series Compound Amount Factor

*P/G* factor: Arithmetic gradient present worth factor

A/G factor: Arithmetic gradient uniform-series factor